Three grades of coherence for non-Archimedean preferences

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in collaboration with




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OUTLINE

1. We review de Finetti's theory of coherence for a (weakly-ordered) preference relation $\preccurlyeq$ over (bounded) real-valued random variables.

- **de Finetti provides a representation of coherent$_1$, weak-ordered preferences using finitely additive, real-valued probabilities and expectations.**
- ***Coherence*$_1$ requires strict preference for a uniformly dominating variable.**

  If for some $\varepsilon > 0$ and for each state ω, $X(\omega) + \varepsilon < Y(\omega)$ then $X \prec Y$.

2. We examine two stronger coherence criteria: coherence$_2$ and coherence$_3$.

- **Coherence$_2$ requires strict preference for a simply dominating variable.**

  If for each state ω, $X(\omega) < Y(\omega)$ then $X \prec Y$.

- **Coherence$_3$ requires strict preference for a weakly dominating variable (*admissibility*).**

  If for each ω, $X(\omega) \leq Y(\omega)$, and for some ω, $X(\omega) < Y(\omega)$ then $X \prec Y$.

- **We review how the two stronger coherence requirements (coherence$_2$ and coherence$_3$) impose unacceptable restrictions on *real-valued* preferences, as seen through the respective representations in terms of real-valued probabilities.**

3. **We accommodate each of the three coherence conditions using (weakly ordered) non-Archimedean preferences.**

- **The central result is that each of these coherent weak-ordered preferences, over (even unbounded) real-valued random variables is represented using non-standard probability and utility.**

- **Simple examples illustrate that coherent$_2$ and coherent$_3$ weak orders cannot be represented using lexicographic probabilities and lexicographic utilities.**

- **Coherence$_3$ supports conditional probability and, more generally, conditional expectations derived entirely from unconditional preferences.**

- **Last, the same approach extends to represent non-Archimedean coherent *strict partial orders* – yielding a version of non-standard IP theory.**

1 – A short review of deFinetti's theory of coherent$_1$ wagering.

We have a zero-sum (sequential) game played between

a *Bookie* and a *Gambler*, with a *Moderator* supervising.

Let $X : \Omega \to \Re$ be a (bounded) real-valued variable defined on a space Ω of possibilities, a space that is well defined for all three players by the Moderator.

The *Bookie*'s *prevision* $p(X)$ on the r.v. X has the operational content that, when the *Gambler* fixes a real-valued quantity $\alpha_{X,\, p(X)}$

then in state ω the resulting payoff to the *Bookie* is $\alpha_{X,\, p(X)}\,[\, X(\omega) - p(X)\,]$

with the opposite payoff to the Gambler.

- **Given X, the *Bookie* offers a *fair price* (a constant variable) $p(X)$ that makes the following two variables indifferent: $X \approx p(X)$**

A simple version of deFinetti's *Book* game proceeds as follows:

1. **The *Moderator* identifies a (possibly infinite) set of random variables $\{X_i\}$**

2. **The *Bookie* announces a prevision $p_i = p(X_i)$ for each r.v. in the set.**

3. **The *Gambler* then chooses (*finitely many*) non-zero terms $\alpha_i = \alpha_{X_i, p(X_i)}$.**

4. **A state $\omega \in \Omega$ is realized by *Nature*: write $X_i(\omega) = X_i$**

5. **The *Moderator* settles up each contract and awards the Bookie (Gambler) the respective SUM of his/her payoffs:**

Total payoff to Bookie $= \sum_{i=1}^{n} \alpha_i [X_i - p_i]$.

Total payoff to Gambler $= -\sum_{i=1}^{n} \alpha_i [X_i - p_i]$.

***Definition*:**

The *Bookie*'s previsions are *incoherent*$_1$ if the *Gambler* can choose finitely many non-zero terms, α_i that assures her/him a (*uniformly*) positive payoff, regardless which state in Ω obtains – and then the *Bookie* loses for sure.

A set of previsions is *coherent*$_1$, if not incoherent$_1$.

***Theorem* (deFinetti):**

A set of previsions is coherent$_1$ *if and only if*

each prevision $p(X)$ is the expectation for X under a common (finitely additive) probability P.

That is, $$p(X) = E_{P(\bullet)}[X] = \int_\Omega X \, dP(\bullet)$$

- ***Corollary 1*****: When the random variables are *indicator functions* for events $\{E_i\}$, so that the gambles are simple bets – with the α's then the stakes in a winner-take-all scheme – then**

  the previsions p_i are coherent$_1$ *if and only if* there is a (f.a.) probability P

  where each prevision is the respective probability $p_i = P(E_i)$.

A short interlude

Three limitations to this general approach ***not*** *addressed in this presentation*

[1] ***Dominance*** **$_{(1,\ 2,\ or\ 3)}$ is invalid in the presence of act-state dependence –** ***moral hazard*****.**

Consider the following case of dominance$_{(1,\ 2,\ or\ 3)}$ between two acts.

	ω_1	ω_2
A_1	*3*	*1*
A_2	*4*	*2*

Act A_2 dominates act A_1. However, if there is ***moral hazard*** **– act-state probabilistic dependence, then A_1 may maximize subjective conditional expected utility, not A_2.**

- **Jiji Zhang's "*Subjective Causal Networks …*" addresses this challenge!**

[2] ***Strategic play*** **by the** ***Bookie*** **against the** ***Gambler*** **may result in a failure to elicit the** ***Bookie*****'s degrees of belief. The** ***Bookie*****'s prevision may differ from her/his credence.**

- **Kevin Zollman's "*The Theory of Games …*," is one of several presentations at this workshop that intersect the theme: where strategic action conflicts with epistemic goals.**

[3] The problem of the ***numeraire*** **– state-dependent utilities.**

De Finetti's game is played with real-valued ***unitless*** **outcomes of variables.**
The ***Bookie*****'s previsions may depend on which currency is used to realize variables.**

2 What becomes of de Finetti's theory when

- **coherence$_1$ – uniform dominance**

is strengthened, either to

- **coherence$_2$ – strict dominance,**

or

- **coherence$_3$ – weak dominance (admissibility)?**

Some Answers

- **Coherence$_2$ precludes some intuitive (merely) f.a. expectations that are coherent$_1$. See de Finetti [1972, p. 77, *fn*.].**

***Example* 1: Consider a denumerably infinite state space $\Omega = \{\omega_1, \omega_2, \ldots\}$.**

Let P be the coherent$_1$ (strongly) finitely additive probability that is *uniform* on Ω.

For all integers i and j, $P(\{\omega_i\}) = P(\{\omega_j\})$. So, $P(\{\omega\}) = 0$.

Consider the (bounded) variable $X(\omega_n) = 1/n$. X is bounded: $0 < X \leq 1$.

Then $E_P[X] = 0$ and so, $X \approx 0$.

But for each ω, $X(\omega) > 0$. So X strictly dominates 0 and coherence$_2$ requires $0 \prec X$.

- **Coherence$_3$ precludes a coherent$_2$ probability whenever some possible event is *P*-null. See Shimony [1955].**

***Example* 2: Consider a binary space $\Omega = \{\omega_1, \omega_2\}$.**

Let *P* be the coherent$_2$ probability supported by ω_1: $P(\{\omega_1\}) = 1$ and $P(\{\omega_2\}) = 0$.

Let I_1 be the indicator for ω_1: $I_1(\omega_1) = 1$ and $I_1(\omega_2) = 0$.

But, $E_P[I_1] = 1$ and then $I_1 \approx 1$.

However, I_1 is weakly dominated by the constant variable 1: $I_1(\omega_1) = 1$, $I_1(\omega_2) = 0$

I_1 is *inadmissible* against the constant 1. Coherence$_3$ requires $I_1 \prec 1$.

How to accommodate all three coherence conditions without imposing such restrictive conditions on the probabilities that represent the corresponding uncertainties over Ω?

3. Non-Archimedean, weakly ordered coherent preference.

In de Finetti's theory of coherent previsions,

(independent of which of the three senses of "coherence" is applied)

for each variable X, the *Bookie* is required to offer a fair price – a *prevision* $p(X)$.

A *prevision* $p(X)$ is a *real-valued* *constant* variable that, in the eyes of the *Bookie*, supports a swap, either way, between the variable X and the (constant) variable $p(X)$.

- **Expressed in terms of the *Bookie*'s preferences,** $X \approx p(X)$.

Definition**: A binary relation $\preccurlyeq$ is a *weak order* if it is transitive,**

and each pair of objects are comparable by the relation, i.e., either $X \preccurlyeq Y$ or $Y \preccurlyeq X$.

Definition**: A binary relation $\preccurlyeq$ on a set $\gimel$ is a <u>*Archimedean*</u> – it admits a (real) Utility representation – if there exists a (real-valued) function $U: \gimel \rightarrow \Re$ where, for $X, Y \in \gimel$**

$$X \preccurlyeq Y \quad \textit{if and only if} \quad U(X) \leq U(Y).$$

Given a real number c, let $C(\omega) = c$ denote the corresponding constant variable.

Under each of the three dominance conditions, when $c < d$, then $C \prec D$.

Let $\preccurlyeq$ be a *coherent*, weakly ordered preference over (bounded) variables defined on a common state-space Ω. Because of the requirements for trading, under coherence:

- **If for each variable X there is a real-valued prevision, $p(X)$, then $\preccurlyeq$ is *Archimedean***

  with $\quad U(X) = p(X),$

  and for real α and β $\quad U(\alpha X + \beta Y) = \alpha U(X) + \beta U(Y) = \alpha p(X) + \beta p(Y).$

We modify de Finetti's prevision-game by *not requiring* that the *Bookie* holds a real-valued (constant) prevision, $p(X)$, for swapping with variable X.

- ***Aside*: We do not require that the variables are bounded.**

Instead, we require that the *Bookie* has a coherent$_{(1,2,\text{ or }3)}$ *weakly ordered* preference $\preccurlyeq$ over the set of real-valued variables defined on a state-space Ω, and where the *Bookie* accepts permitted trades just as in de Finetti's *Prevision Game*.

Coherence$_{(1,2,\text{ or }3)}$ of $\preccurlyeq$ means that the corresponding dominance$_{(1,\,2,\text{ or }3)}$ condition is respected and that the following *Independence* condition is satisfied.

Let a and b be real numbers, c a positive real number, and Y a variable.

$$X_1 \approx X_2 \quad \textit{if and only if} \quad aX_1 + bY \approx aX_2 + bY$$

$$X_1 \prec X_2 \quad \textit{if and only if} \quad cX_1 + bY \prec cX_2 + bY$$

The weak order $\preccurlyeq$ is operationalized through allowed trades,

with the *Bookie* specifying:

(1) the strict partial order $X \prec Y$, which identifies all the 1-way trades.

Where the *Bookie* is willing to swap X for Y, but not vice versa.

(2) the equivalence relation, $X \approx Y$, which identifies all the 2-way trades:

Where the *Bookie* is willing to swap X for Y, and willing to swap Y for X.

Two illustrations of non-Archimedean, coherent$_3$ weak orders

***Example 3*: Let $\Omega = \{\omega_1, \omega_2\}$. Variable X_i is the ordered pair $\langle x_{i1}, x_{i2} \rangle$, where $X_i(\omega_j) = x_{ij}$.**

Define the coherent$_3$ $\preccurlyeq$ weak order where

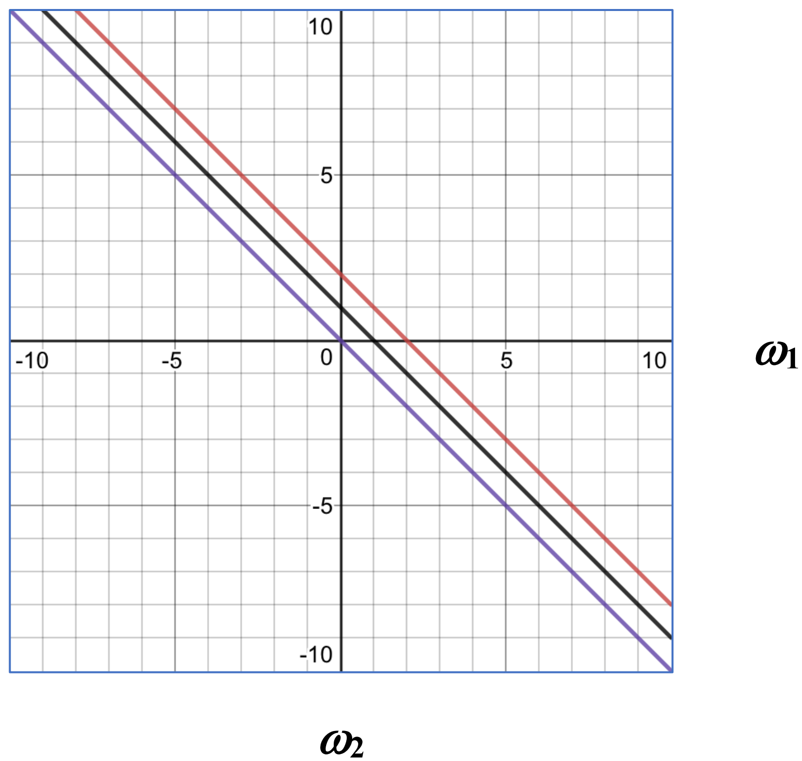
(1) $X_1 \prec X_2$ *iff* $x_{11} + x_{12} < x_{21} + x_{22}$

or $x_{11} + x_{12} = x_{21} + x_{22}$ **and** $x_{11} < x_{21}$.

(2) $X_1 \approx X_2$ *iff* $X_1 = X_2$

Increasing preference for all points on lines to the NE across different lines.

Increasing preference for points to the SE on a given line.



Example 4 **(de Finetti's *Example 1* continued):**

Consider, again, the denumerable state space $\Omega = \{\omega_1, \omega_2, \ldots\}$.

A coherent$_3$ preference allows indifference between state-indicators, $I(\omega_i) \approx I(\omega_j)$.

Each is strictly more desirable than 0 (by dominance$_3$), $0 \prec I(\omega_i)$.

And by dominance$_2$, each is strictly less desirable than $X(\omega_n) = 1/n$, $I(\omega_j) \prec X$,

By coherence$_3$, X is less desirable than an arbitrary positive constant $C(\omega) = c > 0$.

$$0 \prec I(\omega_i) \approx I(\omega_j) \prec X \prec C.$$

The following shows that the two weak orders of Examples 3 and 4 are non-Archimedean.

***Definition*: Let $\rhd$ be a total order on a set $\aleph$. A subset $\beth \subseteq \aleph$ is <u>$\rhd$-*order dense*</u> if, whenever $x \rhd y$ with $x \notin \beth$ and $y \notin \beth$, then there exists $z \in \beth$ with $x \rhd z \rhd y$.**

***Lemma* (Fishburn, 1972):**

Let $\rhd$ be a total order on a set $\aleph$. $\rhd$ is an Archimedean order (i.e., $\rhd$ has a real-valued Utility representation) *if and only if* there is a denumerable $\rhd$-order dense subset $\beth \subseteq \aleph$.

***Recall*: When $\preccurlyeq$ is a weak order, then $\rhd = \preccurlyeq/\approx$ is a total order.**

Next we see that there are no denumerable order dense subsets either in Example 3 or 4. The reasoning is similar in both examples.

Let ℶ be an ⊳-order dense subset and consider the continuum-many reals, 0 < *c* < 1.

In *Example* 3, each *c* denotes a distinct set (a line with slope -1) of continuum-many variables X_α^c where $x_{\alpha,1}^c + x_{\alpha,2}^c = c$.

Since $X_\alpha^c \prec X_b^c$ whenever $x_{\alpha,1}^c < x_{b,1}^c$, if ℶ is an ⊳-order dense subset,

then ℶ contains at least one point from line *c*.

Hence ℶ is uncountable.

In *Example* 4, for each pair of real numbers 0 < *c* < *d* < 1, the ≼ order satisfies

$$0 \prec C \prec C + I(\omega_1) \prec D \prec D + I(\omega_1).$$

If ℶ is an ⊳-order dense subset, then for each 0 < *c* < 1, ℶ contains at least one variable, X_c, where $C \preccurlyeq X_c \preccurlyeq C + I(\omega_1)$.

Hence ℶ is uncountable.

Representing non-Archimedean, coherent weak-orders using non-standard utilities

Let Ξ be a linear space of (standard) real-valued variables on a set Ω.

Let *X* and *Y* belong to Ξ, and let *a* and *b* be (standard) real numbers.

Denote the non-standard real numbers by $^*\Re$.

Aside: **We use the ultra-product model of $^*\Re$.**

Fix one of the three senses of *dominance*.

***Defn*: A non-standard value function *U*: $\Xi \to {}^*\Re$ is a *positive linear functional* if**

- **whenever *Y dominates X* then *U*(X) < *U*(Y)**
- **$U(aX + bY) = aU(X) + bU(Y)$**

- ***Main Theorem*: Let ≼ be a weak order over Ξ.**

≼ is coherent$_{(1,\ 2,\ \text{or}\ 3)}$ *if and only if*

there is a positive linear functional *U* that represents ≼.

Non-standard Probability defined by non-standard Utility.

In the *Main Theorem*, without loss of generality, one may "standardize" the non-standard utility U so that $U(0) = 0$ and $U(1) = 1$.

Then, as U is a positive linear functional, when restricted to indicator variables, that is, with I_E the indicator function for the event E ($\subseteq \Omega$) then

- **U is a (finitely additive) non-standard probability, $U = {}^*P$.**

Let E and F be disjoint subsets of Ω, with $G = E \cup F$.

Then U satisfies:

$$U(I_\Omega) = 1 \text{ and } U(I_\varnothing) = 0$$

$$0 \leq U(I_E) \leq 1$$

$$U(I_G) = U(I_E) + U(I_F)$$

***Definition*:** **Call a non-standard ε a *positive infinitesimal* if**

$0 < \varepsilon < a$ for each positive (standard) real number *a*.

***Example 2* (continued):**

Let $\Omega = \{\omega_1, \omega_2\}$. Variable X_i is the ordered pair $<x_{i1}, x_{i2}>$, where $X_i(\omega_j) = x_{ij}$.

Define the coherent$_3$ $\preccurlyeq$ weak order where

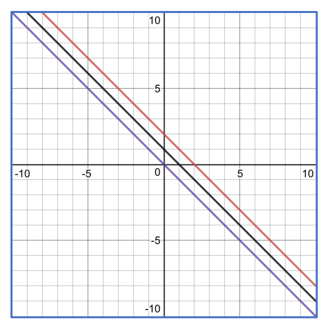
(1) $X_1 \prec X_2$ ***iff*** $x_{11} + x_{12} < x_{21} + x_{22}$

or $x_{11} + x_{12} = x_{21} + x_{22}$ **and** $x_{11} < x_{21}$ **.**

(2) $X_1 \approx X_2$ ***iff*** $X_1 = X_2$**.**

Increasing preference for all points on lines to the NE across different lines.

Increasing preference for points to the SE on a given



ω_1

ω_2

The representing non-standard $^*P$-probability satisfies

$$^*P(I_{\omega_1}) = ½ + \varepsilon, \text{ and } ^*P(I_{\omega_2}) = ½ - \varepsilon$$

for a positive infinitesimal ε.

Some coherent preferences that have no lexicographic-probability representations.

Let $\vec{P} = <P_1, P_2, \ldots>$ be a well-ordered sequence of real-valued probabilities.
Each probability P_i is defined on an algebra $\mathcal{A}$ of subsets of a state-space Ω.

For events E, F that belong to $\mathcal{A}$, say that

$$E \rhd_{\vec{P}} F \text{ iff } P_i(E) < P_i(F) \text{ for the least index } i \text{ where } P_i(E) \neq P_i(F)$$

and $E \approx_{\vec{P}} F$ ***iff*** **for all i, $P_i(E) = P_i(F)$.**

Write $E \unrhd_{\vec{P}} F$ to abbreviate $E \rhd_{\vec{P}} F$ or $E \approx_{\vec{P}} F$.

- **These relations define a qualitative (non-Archimedean) probability.**

(1) $\unrhd_{\vec{P}}$ **is a weak-order**

(2) $0 \unrhd_{\vec{P}} E \unrhd_{\vec{P}} 1$

(3) $E \unrhd_{\vec{P}} F$ ***if and only if*** $E \cup G \unrhd_{\vec{P}} F \cup G$ **whenever** $E \cap G = F \cap G = \emptyset$**.**

***Example* 5: Let Ω be an infinite set. Consider a coherent$_3$ weak-order $\preccurlyeq$ where**

$$I_{\omega_\alpha} \approx I_{\omega_\beta} \quad \text{for each pair of states in } \Omega, \qquad [1]$$

and (by *admissibility*) $$I_\varnothing \prec I_{\omega_\alpha}. \qquad [2]$$

Necessary for a lex-prob $\vec{P}$ to agree with [1], $P_i(\{\omega\}) = 0$ for all i and all $\omega \in \Omega$.
But then $I_\varnothing \approx_{\vec{P}} I_{\omega_\alpha}$, contrary to what [2] requires.

Also, there are coherent$_2$ weak-orders which are not coherent$_3$ and which are not represented by a Utility based on a lex-probability.

***Example 6* (de Finetti's *Example* 1, modified) Consider the denumerable $\Omega = \{\omega_1, \omega_2, \ldots\}$.**
A coherent$_2$ preference allows indifference between each two state-indicators, $I(\omega_m)$ and $I(\omega_n)$, and also indifference between each state indicator and 0,

$$0 \approx I(\omega_m) \approx I(\omega_n) .$$

But by coherence$_2$ (by strict dominance) each of these variables is strictly less desirable than $X(\omega_n) = 1/n$, which yields

$$0 \approx I(\omega_m) \approx I(\omega_n) \prec X .$$

Necessary for a lex-prob $\vec{P}$ to agree with these indifferences is that $P_i(\{\omega_m\}) = 0$ for all i and all m. But then for each i, the expectation $E_{P_i}[X] = 0$, and so $0 \approx_{\vec{P}} X$, contrary to strict dominance (coherence$_2$) which requires $0 \prec X$.

On conditional expectations:

Reconsider de Finetti's *Prevision Game*

***Definition*: A *called-off prevision* $p(X \,||\, E)$ for X,**
made by the *Bookie* has a payoff scheme to the Bookie*:*

$$\alpha_{X||E}\, E(\omega)[\, X(\omega) - p(X \,||\, E)\,].$$

***Corollary 2 to de Finetti's Coherence Theorem*:**

A called-off prevision $p(X \,||\, E)$ is coherent$_1$
alongside the (coherent$_1$) previsions $p(X)$ for X, and when $p(E) > 0$,
if and only if
$p(X \,||\, E)$ is the *conditional expectation* under P for X, given E.

That is, $$p(X \,||\, E) = E_{P(\bullet\, |E)}[X] = \int_\Omega X\, dP(\bullet \,|\, E)\,.$$

But, when $p(E) = 0$, $p(X \,||\, E)$ is unconstrained by coherence$_1$ or coherence$_2$.

The situation is different with coherence$_3$.

Coherence$_3$ entails that if $E \neq \emptyset$ then $0 \prec E$.

Using the non-standard representation of the *Main Theorem*:

If $E \neq \emptyset$ then $0 < {}^*P(F)$.

Then, just as in the real-valued theory given an event E of positive probability, *called-off* preference fixes conditional *probability, given E.

That is, $${}^*P(F \mid E) = {}^*P(F \cap E) \,/\, {}^*P(E).$$

An introduction to *Imprecise non-standard Probability*: *I*P*-theory.

For a final theme, consider another de Finetti result, which addresses extending coherent$_1$ previsions to a larger collection of variables.

Background: The ***Bookie*** assigns coherent$_1$ previsions to a set χ of (bounded) variables.

For each $X \in \chi$ the ***Bookie*** assigns a prevision $p(X)$, and these are coherent$_1$.

By the rules of the ***Prevision Game***, these previsions determine a unique coherent$_1$ previsions for each variable Y in the ***Linear Span***$[\chi]$.

Let Z be a (bounded) variable defined on Ω but outside the ***Linear Span***$[\chi]$.

Define: $\underline{Z} = \{X: X(\omega) \leq Z(\omega)$ and X in the ***Linear Span***$[\chi]\}$ – to approximate from below.

$\overline{Z} = \{X: X(\omega) \geq Z(\omega)$ and X in the ***Linear Span***$[\chi]$ – to approximate from above.

Let $\underline{p}(Z) = \sup_{X \in \underline{Z}} p(X)$ and $\overline{p}(Z) = \inf_{X \in \overline{Z}} p(X)$

Fundamental Theorem of Previsions

To remain coherent$_1$, $p(Z)$ may be any value in the closed interval

$$[\ \underline{p}(Z),\ \ \overline{p}(Z)\ ].$$

Adapting the *Fundamental Theorem* to I*P theory using a *coherent pre-order*.

A *coherent pre-order* is operationalized by a partition into 4 categories of trades:

(1) a strict partial order $X \prec Y$, which identifies all the 1-way trades.
Where the *Bookie* is willing to swap X for Y, but not vice versa.

(2) an equivalence relation, $X \approx Y$, which identifies all the 2-way trades:
Where the *Bookie* is willing to swap X for Y, and willing to swap Y for X.

(3) the *one-way limited* trades, $X \precsim Y$,
This identifies those pairs where the *Bookie* is willing to swap one way, Y for X, but has not resolved whether to trade the other way, X for Y.

(4) the non-comparable pairs, $X \not\Leftrightarrow Y$, where the *Bookie* is unwilling to trade either way.

Note: When both categories (3) and (4) are empty, the pre-order is a weak-order.

In order to be *coherent*$_{(1,\ 2,\ \text{or}\ 3)}$ the pre-order respects the (respective) dominance condition and satisfies the *Independence* condition for allowed trades.

Let a and b be real numbers, c a positive real number, and Y a variable.

$$X_1 \approx X_2 \quad \textit{if and only if} \quad aX_1 + bY \approx aX_2 + bY$$

$$X_1 \prec X_2 \quad \textit{if and only if} \quad cX_1 + bY \prec cX_2 + bY$$

A *coherent extension* of a pre-order preserves all the binary comparisons already fixed by categories (1) and (2), and

either moves some comparisons from category (3) into category (1) or (2),

or moves some comparisons from category (4) into category (1), (2), or (3)

while satisfying the (respective) *Coherence* and *Independence* conditions.

We show how to represent a coherent pre-order by the set of all its coherent, weak-order extensions – the extensions where categories (3) and (4) are empty.

Then, by the *Main Theorem*, each coherent pre-order is represented by the set of non-standard utilities *U* that correspond to each coherent weak-order extension.

- ***Each coherent pre-order is represented as an I*P set of non-standard utilities.***

Summary

We generalize de Finetti's theory of

coherent$_1$, real-valued previsions over (bounded) real-valued variables

to a theory of

coherent$_{(1,2,\text{ or }3)}$ non-Archimedean, weak-orders over real-valued variables.

1 This theory accommodates stronger dominance conditions without having to impose overly restrictive conditions on personal probabilities.

2 A non-standard Utility represents a coherent$_{(1,2,\text{ or }3)}$ non-Archimedean weak-order. The Utility reduces to a non-standard probability over events (indicator variables).

3 Not all non-Archimedean weak orders are representable using lex-probabilities.

4 With coherence$_3$, all conditional probabilities are fixed by unconditional probabilities.

5 We adapt de Finetti's *Fundamental Theorem* to apply to coherent pre-orders, which opens the door to *Imprecise non-standard Probability theory*: I*P- theory